The Deduction of Final Formulas for the Algebraic Solution of the Quartic Equation.

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I.—The solution of a quartic equation depends upon that of a cubic resolvent. Let this cubic be of the form

$$y^3 + 3py^2 + 3qy + r = 0, \tag{1}$$

the three roots of which are expressed by

$$\begin{aligned} y_1 &= -p + s + t, \\ y_2 &= -p + es + e^2t, \\ y_3 &= -p + e^2s + et, \end{aligned}$$

in which e is an imaginary cube root of unity. To find s and t, let h and k first be determined from the given coefficients by

$$h = \frac{1}{2}(-2p^3 + 3pq - r), \quad k = (p^2 - q)^3, \tag{2}$$

and then

$$s = (h + \sqrt{h^2 - k})^{\frac{1}{3}}, \qquad t = (h - \sqrt{h^2 - k})^{\frac{1}{3}}. \tag{3}$$

These formulas are well known; if $p = 0$, that for y_1 becomes the formula of Cardan.

When the coefficients in (1) have such values that $h^2 - k$ is a positive quantity the cubic has one real and two imaginary roots, the values of s and t are real, and the real root is given by y_1. The imaginary roots will then be expressed in the simplest practical form by inserting the values of e and e^2. Thus, the roots are,

$$\left.\begin{aligned} y_1 &= -p + (s + t), \\ y_2 &= -p - \frac{1}{2}(s + t) + \frac{1}{2}(s - t)\sqrt{-3}, \\ y_3 &= -p - \frac{1}{2}(s + t) - \frac{1}{2}(s - t)\sqrt{-3}, \end{aligned}\right\} \tag{4}$$

and these together with (2) and (3) are the final formulas for the algebraic solution of (1).

When $h^2 - k$ is negative the above formulas also algebraically represent the roots, but the numerical solution is said to fail, or more strictly it should be said that the algebraic formulas fail to give numerical results in as simple forms as desired. In this "irreducible case" the values of s and t are imaginary, but $s+t$ is real and so is $(s-t)\sqrt{-3}$, yet their final numerical values cannot be ascertained by algebraic operations. They can be found graphically by trisecting an angle, or trigonometrically by the use of a table of cosines, but the discussion here is concerned only with algebraic solutions.

II.—The cubic resolvent deduced by Euler, in 1770, for the solution of the quartic equation, is one of the simplest and is hence most frequently quoted. Let the proposed quartic be

$$z^4 + 6Bz^2 + 4Cz + D = 0, \tag{5}$$

and let there be taken the cubic resolvent,

$$y^3 + 3By^2 + \frac{1}{4}(9B^2 - D)y - \frac{1}{4}C^2 = 0, \tag{6}$$

whose roots are y_1, y_2, and y_3. Then the roots of the given quartic are, if C is negative,

$$\left.\begin{aligned} z_1 &= +\sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3}, \\ z_2 &= +\sqrt{y_1} - \sqrt{y_2} - \sqrt{y_3}, \\ z_3 &= -\sqrt{y_1} + \sqrt{y_2} - \sqrt{y_3}, \\ z_4 &= -\sqrt{y_1} - \sqrt{y_2} + \sqrt{y_3}, \end{aligned}\right\} \tag{7}$$

and if C is positive the signs before all the radicals are to be reversed.

The reasoning by which the above is deduced need not be given here; it will be found in many of the mathematical reference books as well as in standard treatises. In applying these formulas, however, to numerical solutions, difficulties are found to arise even when the cubic (6) does not fall under the irreducible case. The numerical example generally given is that stated in the article Algebra in the last edition of the Encyclopædia Britannica, namely $z^4 - 25z^2 + 60z - 36 = 0$, which leads to the cubic $y^3 - \frac{25}{2}y^2 + \frac{769}{16}y - \frac{225}{4} = 0$. The roots of this are found "by the rules for cubics," to be $\frac{9}{4}$, $\frac{16}{4}$ and $\frac{25}{4}$, so that $\sqrt{y_1} = \frac{3}{2}$,

$\sqrt{y_2}=\frac{4}{2}$, $\sqrt{y_3}=\frac{5}{2}$; whence the roots of the quartic are $z_1=-6$, $z_2=+3$, $z_3=+2$, $z_4=+1$. Now this example has no place in the exemplification of the algebraic solution, for it falls under the irreducible case where the values of the roots of the cubic resolvent cannot be algebraically obtained.

An examination of numerous authorities has been made, but not in a single instance has there been found a numerical example of legitimate algebraic solution by Euler's formulas. The solvable case, numerically, is that where the proposed quartic has two real and two imaginary roots. Such a quartic leads to a cubic resolvent having one real and two imaginary roots, say y_1 real, and y_2 and y_3 imaginary. The square roots of y_2 and y_3 are also imaginary, and hence z_1, z_2, z_3 and z_4 appear in unmanageable imaginary form, although two of these are real quantities. This, apparently, is the reason why legitimate numerical solutions are not found in connection with Euler's resolvent, although this is better adapted than any other resolvent to the exhibition of the roots of the quartic.

III.—In order to discuss the quartic I prefer to use the complete equation

$$x^4+4ax^3+6bx^2+4cx+d=0. \tag{8}$$

In this, let x be replaced by $z-a$, and it reduces to (5), or,

$$z^4+6Bz^2+4Cz+D=0$$

if the coefficients have the values

$$\left.\begin{aligned} B&=-a^2+b, \\ C&=-2a^3+3ab-c, \\ D&=-3a^4+6a^2b-4ac+d. \end{aligned}\right\} \tag{9}$$

Inserting these in Euler's resolvent (6), it becomes (1), namely,

$$y^3+3py^2+3qy+r=0,$$

in which p, q and r represent the quantities

$$\left.\begin{aligned} p&=-(a^2-b), \\ q&=(a^2-b)^2+\frac{1}{12}(4ac-3b^2-d), \\ r&=-\frac{1}{4}(2a^3-3ab+c)^2. \end{aligned}\right\} \tag{10}$$

The formulas for the roots of this cubic resolvent are given by (4) and need not be written here again. Then the roots of the quartic are, when $2a^3 - 3ab + c$ is negative,

$$\left.\begin{aligned} x_1 &= -a + \sqrt{y_1} + (\sqrt{y_2} + \sqrt{y_3}), \\ x_2 &= -a + \sqrt{y_1} - (\sqrt{y_2} + \sqrt{y_3}), \\ x_3 &= -a - \sqrt{y_1} + (\sqrt{y_2} - \sqrt{y_3}), \\ x_4 &= -a - \sqrt{y_1} - (\sqrt{y_2} - \sqrt{y_3}), \end{aligned}\right\} \tag{11}$$

and when $2a^3 - 3ab + c$ is positive the signs before $\sqrt{y_1}$ and the parentheses are to be reversed.

Now to bring these roots into manageable form for numerical solutions let y_1 be regarded as real, and y_2 and y_3 as imaginary, their expressions in terms of s and t being as given in (4). It is clear that both $\sqrt{y_2} + \sqrt{y_3}$ and $\sqrt{y_2} - \sqrt{y_3}$ are real, and in order that their algebraic expressions may be free from imaginaries it will be well to write

$$\left.\begin{aligned} \sqrt{y_2} + \sqrt{y_3} &= \sqrt{y_2 + y_3 + 2\sqrt{y_2 y_3}}, \\ \sqrt{y_2} - \sqrt{y_3} &= \sqrt{y_2 + y_3 - 2\sqrt{y_2 y_3}}. \end{aligned}\right\} \tag{12}$$

Thus the imaginaries disappear, for as y_2 is of the form $\alpha + \beta i$ and y_3 of the form $\alpha - \beta i$, their sum $y_2 + y_3$ is 2α, and their product $y_1 y_2$ is $\alpha^2 + \beta^2$, both real quantities.

To complete the solution it is now only necessary to obtain the values of y_1, $y_2 + y_3$, $y_2 y_3$, in terms of the coefficients of the given quartic. Let $y_1 = u$, $y_2 + y_3 = v$, and $4y_1 y_2 = w$. Then from (4) are obtained

$$\left.\begin{aligned} y_1 &= -p + s + t = u, \\ y_2 + y_3 &= -2p - s - t = v, \\ 4y_2 y_3 &= (2p + s + t)^2 + 3(s - t)^2 = w, \end{aligned}\right\} \tag{13}$$

in which $-p$ is $a^2 - b$. The quantities s and t are, by (3),

$$s = (h + \sqrt{h^2 - k})^{\frac{1}{3}}, \qquad t = (h - \sqrt{h^2 - k})^{\frac{1}{3}},$$

and h and k are found by substituting in (2) the values of p, q and r as given by (10), whence

$$\left.\begin{aligned} h &= \frac{1}{8}(a^2 d + b^3 + c^2 - 2abc - bd), \\ k &= \frac{1}{64}(b^2 + \frac{1}{3}d - \frac{4}{3}ac)^3. \end{aligned}\right\} \tag{14}$$

Thus x_1, x_2, x_3, x_4 are rationally expressed in terms of a, b, c, d by (11), (12), (13), (2) and (14). A further slight simplification results by putting $h = \frac{1}{8}m$ and $k = \frac{1}{64}n$.

IV.—The following, therefore, are final formulas for the algebraic solution of the quartic equation

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0. \tag{15}$$

First, let m and n be determined from

$$\left.\begin{aligned} m &= a^2d + b^3 + c^2 - 2abc - bd, \\ n &= (b^2 + \tfrac{1}{3}d - \tfrac{4}{3}ac)^3. \end{aligned}\right\} \tag{16}$$

Secondly, let s and t be obtained by

$$\left.\begin{aligned} s &= \tfrac{1}{2}(m + \sqrt{m^2 - n})^{\frac{1}{3}}, \\ t &= \tfrac{1}{2}(m - \sqrt{m^2 - n})^{\frac{1}{3}}. \end{aligned}\right\} \tag{17}$$

Thirdly, let u, v and w be found from

$$\left.\begin{aligned} u &= (a^2 - b) + (s + t), \\ v &= 2(a^2 - b) - (s + t), \\ w &= v^2 + 3(s - t)^2. \end{aligned}\right\} \tag{18}$$

Then the four roots of the quartic equation are given by the expressions

$$\left.\begin{aligned} x_1 &= -a + \sqrt{u} + \sqrt{v + \sqrt{w}}, \\ x_2 &= -a + \sqrt{u} - \sqrt{v + \sqrt{w}}, \\ x_3 &= -a - \sqrt{u} + \sqrt{v - \sqrt{w}}, \\ x_4 &= -a - \sqrt{u} - \sqrt{v - \sqrt{w}}, \end{aligned}\right\} \tag{19}$$

in which the signs before the square roots are to be used as written provided $2a^3 - 3ab + c$ is a negative quantity; but if this is positive all radicals except $\sqrt{w}$ are to be reversed in sign.

V.—As a numerical example let the proposed quartic equation be

$$x^4 + 3x^3 + x^2 - 7x - 30 = 0.$$

Here, by comparison with the general form in (15),

$$a = +\frac{3}{4},\quad b = +\frac{1}{6},\quad c = -\frac{7}{4},\quad d = -30.$$

First, by the use (16), are derived the values

$$m = -\frac{226}{27} \quad\text{and}\quad n = -\frac{405224}{729}.$$

Secondly, from (17) are computed,

$$s = +1.2767 \quad\text{and}\quad t = -1.6100.$$

Thirdly, by (18) are found

$$u = +0.0625,\quad v = +1.125,\quad w = +26.265.$$

Then, as $2a^3 - 3ab + c$ is a negative number, the formulas (19) furnish

$$x_1 = -0.75 + 0.25 + \sqrt{1.125 + 5.125} = +2,$$
$$x_2 = -0.75 + 0.25 - \sqrt{1.125 + 5.125} = -3,$$
$$x_3 = -0.75 - 0.25 + \sqrt{1.125 - 5.125} = -1 + 2i,$$
$$x_4 = -0.75 - 0.25 + \sqrt{1.125 - 5.125} = -1 - 2i,$$

and each of these exactly satisfies the proposed equation.

As a second example let the given quartic be $x^4 + 7x + 6$. Here $a = 0$, $b = 0$, $c = +\frac{7}{4}$, $d = +6$. First, $m = +3\frac{1}{16}$ and $n = +8$. Secondly, $s = +0.8091$ and $t = +0.6180$. Thirdly, $u = +1.427$, $v = -1.427$, and $w = +2.146$. Then, as c is positive, the formulas (19) furnish the roots

$$x_1 = -1.194 - \sqrt{-1.427 + 1.465} = -1.388,$$
$$x_2 = -1.194 + \sqrt{-1.427 + 1.465} = -1.000,$$
$$x_3 = +1.194 - \sqrt{-1.427 - 1.465} = +1.194 - 1.701i,$$
$$x_4 = +1.194 + \sqrt{-1.427 - 1.465} = +1.194 + 1.701i,$$

and these closely satisfy the proposed quartic.

For the third example let the equation be one with equal roots, $x^4 - 9x^2 + 4x + 12 = 0$. Here $a = 0$, $b = -\frac{3}{2}$, $c = +1$, $d = +12$, whence $m = +\frac{125}{8}$ and $n = \left(\frac{25}{4}\right)^3$, which gives $m^2 - n = 0$, as is always the case for

equal roots. Next $s = t = +\frac{5}{4}$, from which $u = +4$, $v = +\frac{1}{2}$, and $w = +\frac{1}{4}$. Then, as c is positive,

$$x_1 = -2 - \sqrt{\frac{1}{2} + \frac{1}{2}} = -3,$$

$$x_2 = -2 + \sqrt{\frac{1}{2} + \frac{1}{2}} = -1,$$

$$x_3 = +2 - \sqrt{\frac{1}{2} - \frac{1}{2}} = +2,$$

$$x_4 = +2 + \sqrt{\frac{1}{2} - \frac{1}{2}} = +2,$$

which are the roots of the given quartic.

Lastly, let the equation be one such that $m = 0$, or $x^4 - 30x^2 - 20x + 20 = 0$. Here, from the given coefficients, m is found to be 0, and $n = \left(\frac{95}{3}\right)^3$. Next, $s + t = 0$ and $3(s - t)^2 = -95$, when $u = 5$, $v = 10$ and $w = 5$. Then, as c is negative, the formulas give the roots

$$x_1 = +\sqrt{5} + \sqrt{10 + \sqrt{5}} = -5.734,$$
$$x_2 = +\sqrt{5} - \sqrt{10 + \sqrt{5}} = +1.262,$$
$$x_3 = -\sqrt{5} + \sqrt{10 - \sqrt{5}} = -0.550,$$
$$x_4 = -\sqrt{5} - \sqrt{10 - \sqrt{5}} = +5.022.$$

VI.—The final formulas (16), (17), (18) and (19) furnish the means for the complete discussion of the circumstances under which the algebraic solution is possible numerically, and also of the conditions necessary for the occurrence of equal roots. The coefficients a, b, c and d are, of course, taken as real quantities in this discussion, although algebraically the formulas are valid for imaginary coefficients also.

First, it is clear that the roots can be obtained numerically, whenever $m^2 - n$ in (17) is a positive quantity, that is, when the coefficients are so related that

$$(a^2d + b^3 + c^2 - 2abc - bd)^2 - (b^2 + \frac{1}{3}d - \frac{4}{3}ac)^3$$

is positive. Then s and t are real, as are also u, v and w, and w is greater than v^2. Hence $v + \sqrt{w}$ will be positive and $v - \sqrt{w}$ will be negative. Accordingly

the roots x_1 and x_2 are real and the roots x_3 and x_4 are imaginaries of the form $\alpha \pm \beta i$.

Secondly, the roots of the quartic are easily found whenever $m^2 - n$ is zero, which is well known as the condition for equal roots. For this case s and t are equal, and hence $v = \sqrt{w}$, so that the roots x_3 and x_4 are equal. The formulas (19) then reduce to

$$x_1 = -a + \sqrt{u} + \sqrt{2v},$$
$$x_2 = -a + \sqrt{u} - \sqrt{2v},$$
$$x_3 = x_4 = -a - \sqrt{u},$$

in which $u = a^2 - b + n^{\frac{1}{3}}$ and $v = 2(a^2 - b) - n^{\frac{1}{3}}$, while the signs before the radicals are to be used as before. For two pairs of equal roots the further condition $v = 0$ is necessary, or the coefficients are related so that

$$4(a^2 - b)^3 - (b^2 + \frac{1}{3}d - \frac{4}{3}ac)$$

also vanishes. Then one pair of equal roots is $x_1 = x_2 = -a + \sqrt{3(a^2 - b)}$ and the other is $x_3 = x_4 = -a - \sqrt{3(a^2 - b)}$.

Thirdly, the roots can be computed when $m = 0$, that is when the coefficients are so related that $a^2d + b^3 + c^2 - 2abc - bd = 0$. Then $s + t = 0$, $(s - t)^2 = -n^{\frac{1}{3}}$, and hence $u = a^2 - b$, $v = 2(a^2 - b)$ and $w = v^2 - 3b^2 + d$, whence by (19) the roots are obtained.

The roots of any numerical quartic can be determined when a known relation, other than that expressed by the coefficients, exists between them. For instance, as one of the most important cases, let $x_1x_2 + x_3x_4 = 0$. Then it will be found that the cubic radicals entirely disappear from the general formulas, and that (18) reduce to

$$u = a^2 - \frac{3}{2}b, \quad v = u + a^2, \quad w = 4a(au + c) - d,$$

while the coefficients are connected by the necessary relation $c^2 + ud = 0$. For instance, the quartic $x^4 - 11x^3 + 28x^2 + 36x - 144 = 0$ has its coefficients thus related, and the formulas give the roots $x_1 = +3$, $x_2 = +4$, $x_3 = -6$, and $x_4 = +2$.

When $m^2 - n$ is negative the final formulas lead, in general, to the irreducible case, where the quartic has either four real roots or four imaginary

roots. The formulas here correctly represent the roots, but their numerical expressions cannot be algebraically reduced to simple forms.

VII.—In conclusion it may be observed that the algebraic solution of all critical and special cases of the quartic equation is effected by the final formulas (16), (17), (18), and (19), and that simple expressions for the roots may be deduced for some of these cases.

Perhaps the simplest instance is $x^4 - e^4 = 0$, for which, using the formulas in succession, there is found $x_1 = +e$, $x_2 = -e$, $x_3 = +ei$, and $x_4 = -ei$.

The quartic $x^4 + 6bx^2 + d$ is a special case, the solution of which is easily made by quadratics. The general formulas do not, at first sight, appear to reduce to the same expressions, but by observing that $2a^3 - 3ab + c$ vanishes it is clear that $\sqrt{u} = 0$. Hence $s + t = b$, and $v = -3b$. Then, since $st = \frac{1}{4}n^{\frac{1}{3}}$, there is found $3(s-t)^2 = -d$, so that $w = 9b^2 - d$. Therefore (19) reduce to

$$x_1 = x_2 = \pm\sqrt{-3b + \sqrt{9b^2 - d}},$$
$$x_3 = x_4 = \pm\sqrt{-3b - \sqrt{9b^2 - d}},$$

which are the same as given by the quadratic solution.

If $d = 0$ one of the roots of the quartic vanishes and the formulas for the other three will reduce to those for the solution of the cubic. As a special case let $x^4 - e^3x = 0$, where $a = 0$, $b = 0$, $c = -\frac{1}{4}e^3$, $d = 0$. Then (16) give $m = +\frac{1}{16}e^6$, $n = 0$, whence by (17) there is found $s = +\frac{1}{4}e^2$ and $t = 0$. Next (17) give $u = +\frac{1}{4}e^2$, $v = -\frac{1}{2}e^2$ and $w = +\frac{1}{4}e^4$. Finally formulas (18) furnish the roots $x_1 = +e$, $x_2 = 0$, $x_3 = (-\frac{1}{2} + \frac{1}{2}\sqrt{-3})e$, $x_4 = (-\frac{1}{2} - \frac{1}{2}\sqrt{-3})e$, where the coefficients of e are the cube roots of unity.

Thus the final formulas here deduced give the algebraic solution of the quartic equation for all possible instances, and they lead to ready and exact numerical solutions whenever the proposed quartic does not fall under the irreducible case.

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